Lattice polytopes with a given h^*-polynomial

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Abstract. Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. It is well-known that $h^*_\Delta(t) := (1-t)^{n+1}\sum_{k\geq 0}|k\Delta \cap \mathbb{Z}^n|t^k$ is a polynomial of degree $d \leq n$ with nonnegative integral coefficients. Let $AGL(n,\mathbb{Z})$ be the group of invertible affine integral transformations which naturally acts on $\mathbb{R}^n$. For a given polynomial $h^* \in \mathbb{Z}[t]$, we denote by $C_{h^*}(n)$ the number $AGL(n,\mathbb{Z})$-equivalence classes of n-dimensional lattice polytopes such that $h^* = h^*_\Delta(t)$. In this paper we show that $\{C_{h^*}(n)\}_{n\geq 1}$ is a monotone increasing sequence which eventually becomes constant. This statement follows from a more general combinatorial result whose proof uses methods of commutative algebra. We give an explict description of the sequence $\{C_{h^*}(n)\}_{n\geq 1}$ for some special polynomials h^*.

1. Introduction

Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional lattice polytope, i.e. all vertices of Δ are contained in $\mathbb{Z}^n$. Denote by $\mathrm{vol}(\Delta)$ the usual euclidean volume of Δ. Define the *normalized volume* of Δ to be

$$\mathrm{Vol}_{\mathbb{N}}(\Delta) := n!\mathrm{vol}(\Delta).$$

It is easy to see that $\mathrm{Vol}_{\mathbb{N}}(\Delta)$ is a positive integer. If Δ is a lattice simplex then $\mathrm{Vol}_{\mathbb{N}}(\Delta) \in \mathbb{N}$ follows from a direct calculation (for general case one uses a simplicial subdivision of Δ into lattice simplices). Let us define the *codegree* of Δ as

$$\mathrm{codeg}\,\Delta := \min\{k \in \mathbb{Z}_{\geq 0} \;:\; |\mathrm{Int}(k\Delta) \cap \mathbb{Z}^n| \neq 0\},$$

where $\mathrm{Int}(k\Delta)$ denotes the interior of $k\Delta$. We notice that $\mathrm{codeg}\,\Delta \leq n+1$, because the sum of vertices of any n-dimensional lattice subsimplex $\Delta' \subset \Delta$ is an element of $\mathrm{Int}((n+1)\Delta') \cap \mathbb{Z}^n$. The nonnegative integer

$$\deg \Delta := n + 1 - \mathrm{codeg}\,\Delta$$

will be called *degree* of Δ.

Let $\Pi(\Delta) \subset \mathbb{R}^{n+1}$ be a standard $(n+1)$-dimensional pyramid with height 1 over $\Delta \subset \mathbb{R}^n$, i.e. $\Pi(\Delta)$ is a convex hull of the n-dimensional polytope $\Delta \cong (\Delta, 0) \subset \mathbb{R}^{n+1}$ and the lattice vertex $v = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$.

Since $\mathrm{vol}(\Pi(\Delta)) = \frac{1}{n+1}\mathrm{vol}(\Delta)$, we have

$$\mathrm{Vol}_{\mathbb{N}}(\Pi(\Delta)) = \mathrm{Vol}_{\mathbb{N}}(\Delta).$$

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By intersecting $k\Pi(\Delta) \subset \mathbb{R}^n$ with hyperplanes $x_{n+1} = m$ $(1 \leq m \leq k-1)$ we obtain

$$|\mathrm{Int}(k\Pi(\Delta)) \cap \mathbb{Z}^{n+1}| = \sum_{m=1}^{k-1} |\mathrm{Int}(m\Delta) \cap \mathbb{Z}^n|.$$

The latter implies $\operatorname{codeg} \Pi(\Delta) = \operatorname{codeg} \Delta + 1$ and

$$\deg \Pi(\Delta) = \deg \Delta.$$

Definition 1.1. Let $AGL(n, \mathbb{Z})$ be the group of affine integral linear transformations which naturally acts on $\mathbb{R}^n$. We denote by $C(V, d, n)$ the number of $AGL(n, \mathbb{Z})$-equivalence classes of n-dimensional lattice polytopes such that $\mathrm{Vol}_{\mathbb{N}}\Delta = V$ and $\deg \Delta = d$.

Remark 1.2. The fact that there exists only finitely many $AGL(n, \mathbb{Z})$-equivalence classes of n-dimensional lattice polytopes of fixed volume is classically well-known (see e.g. [**LZ**]).

Now we formulate the main result of the paper:

Theorem 1.3. *Let us fix some integers V and d. Then for*

$$n \geq 4d \binom{2d + V - 1}{2d}$$

every n-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$ with $\mathrm{Vol}_{\mathbb{N}}(\Delta) = V$, $\deg \Delta = d$ is a pyramid $\Pi(\Delta')$ over an $(n-1)$-dimensional lattice polytope $\Delta' \subset \mathbb{R}^{n-1}$.

2. Some applications

Let us discuss some consequences of Theorem 1.3.

Definition 2.1. Let k be a nonnegative integer and Δ be an n-dimensional lattice polytope. We define *k-fold pyramid $\Pi^{(k)}(\Delta)$ over Δ* as follows:
(i) $\Pi^{(0)}(\Delta) := \Delta$;
(ii) $\Pi^{(k+1)}(\Delta) := \Pi(\Pi^{(k)}(\Delta))$.

Corollary 2.2. For fixed integers V, d there exists a positive constant $\nu = \nu(d, V)$ such that for $n > \nu$ every n-dimensional lattice polytope Δ with $\mathrm{Vol}_{\mathbb{N}}(\Delta) = V$, $\deg \Delta = d$ is an $(n - \nu)$-fold pyramid over a ν-dimensional lattice polytope Δ', i.e.

$$\Delta = \Pi^{(n-\nu)}(\Delta').$$

Proof. The statement immediately follows from 1.3 by induction if we set

$$\nu := 4d \binom{2d + V - 1}{2d} - 1.$$

□

Now we show that for fixed integers V and d, one has

$$C(V, d, 1) \leq C(V, d, 2) \leq \ldots \leq C(V, d, n) \leq \ldots.$$

PROPOSITION 2.3. Let k be a nonnegative integer. Two n-dimensional lattice polytopes $\Delta, \Delta' \subset \mathbb{R}^n$ are $AGL(n,\mathbb{Z})$-equivalent if and only if two k-fold pyramids $\Pi^{(k)}(\Delta), \Pi^{(k)}(\Delta') \subset \mathbb{R}^{n+k}$ are $AGL(n+k,\mathbb{Z})$-equivalent.

PROOF. The case $k = 0$ is trivial. If Δ and Δ' are $AGL(n,\mathbb{Z})$-equivalent, then the standard embeddings $\Delta \hookrightarrow \Pi^{(k)}(\Delta)$, $\Delta' \hookrightarrow \Pi^{(k)}(\Delta')$ and $AGL(n,\mathbb{Z}) \hookrightarrow AGL(n+k,\mathbb{Z})$ show that $\Pi^{(k)}(\Delta)$ and $\Pi^{(k)}(\Delta')$ are $AGL(n+k,\mathbb{Z})$-equivalent. Now assume that $\Pi^{(k)}(\Delta)$ and $\Pi^{(k)}(\Delta')$ are $AGL(n+k,\mathbb{Z})$-equivalent. We write

$$\Pi^{(k)}(\Delta) = \mathrm{conv}\{\Delta, v_1, \ldots, v_k\}, \;\; \Pi^{(k)}(\Delta') = \mathrm{conv}\{\Delta', v_1', \ldots, v_k'\}.$$

Let $g \in AGL(n+k,\mathbb{Z})$ be an element such that $g\Pi^{(k)}(\Delta) = \Pi^{(k)}(\Delta')$. If $g\Delta = \Delta'$, then we are done. Otherwise $g\Delta \subset \Pi^{(k)}(\Delta')$ contains a vertex $v_i' \in \Pi^{(k)}(\Delta')$. Therefore, $g\Delta$ is itself a pyramid $\Pi(\Theta)$ over an $(n-1)$-dimensional face $\Theta' \subset \Pi^{(k)}(\Delta')$, i.e. Δ is also a pyramid $\Pi(\Delta_1)$ over an $(n-1)$-dimensional lattice polytope $\Delta_1 := g^{-1}\Theta'$. Analogously $g^{-1}\Delta'$ must contain a vertex $v_j \in \Pi^{(k)}(\Delta)$ and Δ' is a pyramid $\Pi(\Delta_1')$ over an $(n-1)$-dimensional lattice polytope Δ_1'. It remains to show that Δ_1 and Δ_1' are $AGL(n-1,\mathbb{Z})$-equivalent. For this we can apply the same arguments as for Δ and Δ' and come to either $g\Delta_1 = \Delta_1'$, or to another pair of $(n-2)$-dimensional lattice polytopes Δ_2 and Δ_2' such that $\Delta_1 = \Pi(\Delta_2)$ and $\Delta_1' = \Pi(\Delta_2')$. After finitely many steps this process terminates. $\square$

COROLLARY 2.4. The correspondence $\Delta \mapsto \Pi^{(k)}(\Delta)$ is injective on the set of equivalence classes of lattice polytopes of fixed degree and fixed normalized volume considered modulo affine integral linear transformations. For all natural numbers V, d, n, k one has

$$C(V,d,n) \leq C(V,d,n+k),$$

where for sufficiently large n the number $C(V,d,n)$ does not depend on n.

Now consider two power series

$$P(\Delta,t) = \sum_{k \geq 0} |k\Delta \cap \mathbb{Z}^n| t^k, \;\; Q(\Delta,t) = \sum_{k>0} |\mathrm{Int}(k\Delta) \cap \mathbb{Z}^n| t^k.$$

It is well-known that $P(\Delta,t)$ and $Q(\Delta,t)$ are rational functions such that

$$Q(\Delta,t) = (-1)^{n+1} P(\Delta,t^{-1}), \;\; P(\Delta,t) = \frac{h^*_\Delta(t)}{(1-t)^{n+1}},$$

where $h^*_\Delta(t)$ is a polynomial with nonnegative integral coefficients satisfying the conditions

$$h^*_\Delta(0) = 1, \;\; h^*_\Delta(1) = \mathrm{Vol}_{\mathbb{N}}(\Delta).$$

DEFINITION 2.5. We call the polynomial

$$h^*_\Delta(t) := (1-t)^{n+1} P(\Delta,t)$$

h^*-polynomial of the lattice polytope Δ (this definition is inspired by the notion of h^*-vector considered by Stanley in [**St**]).

REMARK 2.6. It follows from the formula

$$Q(\Delta,t) = \frac{t^{n+1} h^*_\Delta(t^{-1})}{(1-t)^{n+1}}$$

that $\operatorname{codeg}\Delta$ equals the multiplicity of the root $t=0$ of the polynomial $t^{d+1}h^*_\Delta(t^{-1})$ and

$$\deg \Delta = \deg h^*_\Delta(t).$$

Since

$$P(\Pi(\Delta), t) = \frac{P(\Delta,t)}{1-t},$$

we have also

$$h^*_\Delta(t) = h^*_{\Pi(\Delta)}(t).$$

Definition 2.7. Let $h^*(t) \in \mathbb{Z}[t]$ be a polynomial with nonnegative integral coefficients. We denote by $C_{h^*}(n)$ the number of $AGL(n,\mathbb{Z})$-equivalence classes of n-dimensional lattice polytopes Δ such that $h^*_\Delta = h^*$.

Since there exist only finitely many polynomials $h^*(t)$ with nonnegative integral coefficients of given degree d and given sum of coefficients $h^*(1) = V$ we obtain

$$C(V,d,n) = \sum_{h^*(1)=V, \deg h^* = d} C_{h^*}(n).$$

Theorem 1.3 together with 2.4 implies the following:

Corollary 2.8. Let $h^*(t) \in \mathbb{Z}[t]$ be a polynomial of degree d. Then for sufficiently large n every n-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$ such that $h^*_\Delta = h^*$ is a pyramid $\Pi(\Delta')$ over a $(n-1)$-dimensional lattice polytope $\Delta' \subset \mathbb{R}^{n-1}$, i.e. one obtains a monotone sequence

$$C_{h^*}(1) \le C_{h^*}(2) \le \cdots \le C_{h^*}(n) \le \cdots$$

which eventually becomes constant.

3. The proof

The proof of Theorem 1.3 uses methods of commutative algebra in the spirit of [**MS**].

Let σ_Δ be the $(n+1)$-dimensional cone in $\mathbb{R}^{n+1}$ over Δ:

$$\sigma_\Delta := \{(r\Delta, r) \in \mathbb{R}^{n+1} \ : \ r \in \mathbb{R}_{\ge 0}\} \subset \mathbb{R}^{n+1}.$$

The set $M_\Delta := \sigma_\Delta \cap \mathbb{Z}^{n+1}$ of all lattice points in the cone σ_Δ is a monoid with respect to sum. Moreover, M_Δ is a *graded* monoid with respect to the $(n+1)$-th coordinate, i.e. the degree of a lattice point $(m,k) \in \mathbb{Z}^n \times \mathbb{Z}$ equals k. We define $S_\Delta := \mathbb{C}[M_\Delta]$ to be the graded semigroup $\mathbb{C}$-algebra of the graded monoid M_Δ. Since the k-th homogeneous component M^k_Δ of M_Δ has form

$$M^k_\Delta = \{(m,k) \in \mathbb{Z}^n \times \mathbb{Z} \ : \ m \in k\Delta\}$$

we have

$$\dim_{\mathbb{C}} S^k_\Delta = |M^k_\Delta| = |k\Delta \cap \mathbb{Z}^n|.$$

This allows to interpret the power series $P(\Delta,t)$ as a Hilbert-Poincaré series of the graded commutative $\mathbb{C}$-algebra $S_\Delta = \bigoplus_{k\ge 0} S^k_\Delta$, i.e.

$$P(\Delta,t) = \sum_{k \ge 0} (\dim_{\mathbb{C}} S^k_\Delta) t^k.$$

We remark that M_Δ is a finitely generated graded monoid, i.e. S_Δ is a graded finitely generated $\mathbb{C}$-algebra.

DEFINITION 3.1. We call a set $\mathcal{X} := \{x_1, \dots, x_l\} \subset \sigma_\Delta \cap \mathbb{Z}^{n+1}$ a *minimal generating set of the monoid* M_Δ if $\mathcal{X}$ generates the monoid M_Δ and every lattice point $x_i \in \mathcal{X}$ cannot be represented as a linear combination of $\mathcal{X} \setminus \{x_i\}$ with nonnegative integral coefficients.

REMARK 3.2. We note that every minimal generating set $\mathcal{X} := \{x_1, \dots, x_l\}$ of M_Δ must contain the set M^1_Δ of all lattice points in σ_Δ of degree 1, because these lattice points cannot be represented as nonnegative integral linear combination of other lattice points in M_Δ. So we have $M^1_\Delta \subset \mathcal{X}$ and $l \geq |M^1_\Delta| = |\Delta \cap \mathbb{Z}^n| \geq n+1$.

DEFINITION 3.3. We associate with each lattice point $x_i \in \mathcal{X}$ a variable X_i and denote by A the polynomial algebra

$$A := \mathbb{C}[X_1, \dots, X_l].$$

The grading of A is defined by the grading of the lattice points in $\mathcal{X}$:

$$\deg X_i := \deg x_i, \quad i = 1, \dots, l.$$

REMARK 3.4. Now the finitely generated $\mathbb{C}$-algebra S_Δ can be written as

$$S_\Delta \cong A/I$$

where I is the homogeneous ideal in A generated by binomials

$$B = B(R) := X_{i_1}^{a_1} \dots X_{i_s}^{a_s} - X_{j_1}^{b_1} \dots X_{j_r}^{b_r},$$

corresponding to linear relations

$$R \ : \ a_1 x_{i_1} + \dots + a_s x_{i_s} = b_1 x_{j_1} + \dots + b_r x_{j_r},$$
$$\{i_1, \dots, i_s\} \cap \{j_1, \dots, j_r\} = \emptyset, \quad a_i, b_j \in \mathbb{Z}_{>0}.$$

The key observation in the proof of Theorem 1.3 is the following statement:

PROPOSITION 3.5. Let $\mathcal{X} := \{x_1, \dots, x_l\}$ be a minimal generating set of M_Δ and let $\{B_1, \dots, B_p\}$ be a set of binomials generating the ideal I. Then $\Delta = \Pi(\Delta')$ for some $(n-1)$-dimensional lattice polytope Δ' if and only if there exists a lattice point $x_i \in M^1_\Delta$ such that the corresponding variable X_i does not appear in any of binomials $B_1, \dots, B_p$. In the latter case x_i is the vertex of the pyramid $\Pi(\Delta')$ and $S_\Delta \cong S_{\Delta'}[X_i]$ is a polynomial ring over $S_{\Delta'}$.

PROOF. "$\Rightarrow$": Let $\Delta = \Pi(\Delta') \subset \mathbb{R}^n$ for some $(n-1)$-dimensional lattice polytope Δ'. We can assume that all points of the polytope Δ' have zero last n-th coordinate and the vertex v of the pyramid $\Pi(\Delta')$ is the lattice point $(0, \dots, 0, 1) \in \mathbb{R}^n$. In this case, all lattice points in the monoid $M_\Delta \subset \mathbb{Z}^{n+1}$ have nonnegative n-th coordinates. Since $(v, 1) \in M^1_\Delta$, by 3.2, we obtain $(v, 1) \in \mathcal{X}$, i.e., $(v, 1) = x_i$ for some $1 \leq i \leq l$. Assume that for some $j \in \{1, \dots, p\}$ the variable X_i corresponding to x_i appears in the binomial

$$B_j = X_{i_1}^{a_1} \dots X_{i_s}^{a_s} - X_{j_1}^{b_1} \dots X_{j_r}^{b_r}.$$

Without loss of generality we can assume that $i = i_1$, $(a_1 > 0)$. Then one has an integral linear relation

$$a_1 x_i + a_2 x_{i_2} + \dots + a_s x_{i_s} = b_1 x_{j_1} + \dots + b_r x_{j_r},$$
$$\{i, i_2, \dots, i_s\} \cap \{j_1, \dots, j_r\} = \emptyset.$$

Since $x_i = (v, 1) \in \mathbb{Z}^{n+1}$ has positive n-th coordinate and all lattice points in $\mathcal{X}$ have nonnegative n-th coordinate, we obtain that there exists $q \in \{j_1, \ldots, j_r\}$ such that x_q also has a positive n-th coordinate. Using the splitting

$$M_\Delta = M_{\Delta'} \oplus \mathbb{Z}x_i,$$

we can write $x_q = x'_q + cx_i$ where $x'_q \in M_{\Delta'}$ and $c > 0$ (all elements of $M_{\Delta'}$ have zero n-th coordinate). We remark that $x'_q \neq 0$ because otherwise we would have a contradiction to minimality of $\mathcal{X}$ or to the condition $\{i, i_2, \ldots, i_s\} \cap \{j_1, \ldots, j_r\} = \emptyset$. Therefore x'_q is a positive linear combination of some lattice points from $\mathcal{X}_0 := \mathcal{X} \cap M_{\Delta'}$. This also contradicts the minimality of $\mathcal{X}$. Thus x_i does not appear in any of binomials $B_1, \ldots, B_p$.

"$\Leftarrow$": Let $x_i \in \mathcal{X}$ be a lattice point such that the corresponding variable X_i does not appear in any of binomials $B_1, \ldots, B_p$. Let M'_Δ be the monoid generated by $\mathcal{X}' := \mathcal{X} \setminus \{x_i\}$. Then $S_\Delta \cong A/(B_1, \ldots, B_p)$ is isomorphic to the polynomial ring $\mathbb{C}[M'_\Delta][X_i]$. In particular, we have

$$\text{Krull dim}\, \mathbb{C}[M'_\Delta] = \text{Krull dim}\, S_\Delta - 1 = n.$$

Therefore all lattice points from $\mathcal{X}'$ belong to a n-dimensional linear subspace $L \subset \mathbb{R}^{n+1}$. The subspace L cuts the cone σ_Δ along its n-dimensional face Θ, because all generators $\mathcal{X}$ of M_Δ except x_i belong to L. Denote by H the affine hyperplane in $\mathbb{R}^{n+1}$ consisting of all points $y \in \mathbb{R}^{n+1}$ whose $(n+1)$-th coordinate equals 1. Then $x_i \in H$, $\Delta = \sigma_\Delta \cap H$ and $\Delta' := \Theta \cap H$ is a convex polytope such that $\Delta = \text{conv}(x_i, \Delta')$. Therefore all vertices of Δ' are lattice points and $M_{\Delta'} = M_\Delta \cap L$. Since there is no nontrivial relations between x_i and $M_{\Delta'}$ we have

$$M_\Delta = M_{\Delta'} \oplus \mathbb{Z}x_i.$$

Thus Δ is a standard pyramid over Δ' with vertex x_i. $\square$

By well-known result of Hochster [**Ho**], S_Δ is a Cohen-Macaulay ring of Krull dimension $n + 1$. We consider a minimal graded free resolution of S_Δ as an A-module:

$$P^\bullet \;\;:\;\; 0 \to P_{l-n-1} \to \cdots \to P_1 \to P_0 \to S_\Delta \to 0$$

where P_i is a free graded A-module of rank α_i. The minimality of the resolution means that all elements of the $\alpha_{i-1} \times \alpha_i$-matrix of the differential $d_i \;:\; P_i \to P_{i-1}$ are contained in the maximal homogeneous ideal μ of A. It is well-known that such a free resolution is uniquely determined up to isomorphism. Moreover, one has

$$\alpha_i = \dim_{\mathbb{C}} \text{Tor}_i^A(\mathbb{C}, S_\Delta)$$

where $\mathbb{C}$ is considered as A-module via the isomorphism $\mathbb{C} \cong A/\mu$. In particular, we have $\alpha_0 = 1$ and α_1 equals the minimal number of generators of the ideal I, i.e. $\alpha_1 = \dim_{\mathbb{C}} I/\mu I$.

Let $\{y_0, y_1, \ldots, y_n\} \subset S^1_\Delta$ be a maximal S_Δ-regular sequence consisting of elements of degree 1. The existence of such a regular sequence follows from the fact that S^1_Δ generates a primary ideal in S_Δ whose radical is the maximal homogeneous ideal in S_Δ. Using the surjective homomorphism $A \to S_\Delta$, we can choose $n + 1$ linearly independent homogeneous linear forms $Y_0, Y_1, \ldots, Y_n \in A = \mathbb{C}[X_1, \ldots, X_l]$ whose images in S_Δ coincide with $y_0, y_1, \ldots, y_n$. Let us define

$$R := S_\Delta/(y_0, y_1, \ldots, y_n), \;\; \overline{I} = I/(Y_0, Y_1, \ldots, Y_n)I,$$

$$\overline{A} := A/(Y_0, Y_1, \ldots, Y_n),\ \overline{\mu} := \mu/(Y_0, Y_1, \ldots, Y_n),$$
$$\overline{P_i} = P_i/(Y_0, Y_1, \ldots, Y_n)P_i,\ \ i = 0, \ldots, l-n-1.$$

PROPOSITION 3.6. The complex $\overline{P^\bullet} = P^\bullet \otimes_A \overline{A}$ is exact so that we can consider

$$\overline{P^\bullet}\ :\ 0 \to \overline{P_{l-n-1}} \to \cdots \to \overline{P_1} \to \overline{P_0} \to R \to 0$$

as a minimal graded free resolution of $R \cong \overline{A}/\overline{I}$ as a $\overline{A}$-module. In particular, the numbers and the degrees of the minimal generators of the ideals $I \subset A$ and $\overline{I} \subset \overline{A}$ are the same.

PROOF. The i-th cohomology of the complex $\overline{P^\bullet} = P^\bullet \otimes_A \overline{A}$ equals

$$\mathrm{Tor}_i^A(\overline{A}, S_\Delta).$$

On the other hand, one can compute $\mathrm{Tor}_i^A(\overline{A}, S_\Delta)$ using the Koszul complex on the elements $Y_0, Y_1, \ldots, Y_n$ which is a graded free resolution of $\overline{A}$ over A. Since the sequence $Y_0, Y_1, \ldots, Y_n$ is S_Δ-regular $\mathrm{Tor}_i^A(\overline{A}, S_\Delta) = 0$ for all $i > 0$ and $\mathrm{Tor}_0^A(\overline{A}, S_\Delta) = R$. Therefore $\overline{P^\bullet}$ is exact. The minimality of the graded free resolution $\overline{P^\bullet}$ follows from the fact that the maximal homogeneous ideal $\overline{\mu} \subset \overline{A}$ is the image under the surjection $A \to \overline{A}$ of the maximal homogeneous ideal $\mu \subset A$. Since

$$P_1/\mu P_1 \cong \overline{P_1}/\overline{\mu}\overline{P_1}$$

we obtain that the numbers and the degrees of the minimal generators of two ideals $I \subset A$ and $\overline{I} \subset \overline{A}$ are the same. □

PROPOSITION 3.7. Let Δ be an n-dimensional lattice polytope such that $V = \mathrm{Vol}_{\mathbb{N}}(\Delta)$. Then the monoid M_Δ contains a minimal generating subset $\mathcal{X}$ containing $l \leq V + n$ elements which all have degree $\leq d$.

PROOF. We remark that the coefficients h_i^* of the h^*-polynomial h_Δ^* are equal to the dimensions of the homogeneous components of the graded artinian ring

$$R = R^0 \oplus R^1 \oplus \cdots \oplus R^d,$$

i.e. $h^i = \dim_{\mathbb{C}} R^i$ and $h_\Delta^*(1) = V$. Since S_Δ is Cohen-Macaulay we obtain that the $\mathbb{C}$-algebra S_Δ is a free module of rank V over the polynomial ring $B := \mathbb{C}[y_0, y_1, \ldots, y_n]$ (S_Δ is an integral extension of $\mathbb{C}[y_0, y_1, \ldots, y_n]$). Consider $\mathcal{Z} := M_\Delta^1 \cup \{z_1, \ldots, z_m\}$ where $z_1, \ldots, z_m$ are some lattice points of degree k $(2 \leq k \leq d)$ in M_Δ such that their images $\overline{z_1} \ldots, \overline{z_m}$ in R form a basis of the $\mathbb{C}$-vector space $\bigoplus_{i=2}^d R^i$. We note that $\mathcal{Z}$ generates the $\mathbb{C}$-algebra S_Δ (and hence also the monoid M_Δ), because the $\mathbb{C}$-algebra generated by $\mathcal{Z}$ contains the polynomial ring B and all generators of S_Δ as a finite B-module. Thus we have

$$|\mathcal{Z}| = |M_\Delta^1| + \sum_{i \geq 2} h_i^*.$$

Since $h_0^* = 1,\ \ h_1^* = |M_\Delta^1| - n - 1$ the generating subset $\mathcal{Z}$ contains $n + \sum_{i \geq 0} h_i^* = n + V$ lattice points and all elements in $\mathcal{Z}$ have degree $\leq d$. □

PROPOSITION 3.8. All binomials in a minimal generating set for the ideal $I \subset A$ have degree at most $2d$.

Proof. By 3.6, it is sufficient to prove the same statement for the ideal $\overline{I} \subset \overline{A}$. Let $\overline{A}^k$ and $\overline{I}^k$ denote k-th homogeneous components of the graded ring $\overline{A}$ and its homogeneous ideal $\overline{I}$. Since for $k \geq d+1$ one has $\overline{A}^k = \overline{I}^k$, it is sufficient to show that for all $i > 0$ the homogeneous component $\overline{A}^{2d+i}$ is generated by products

$$\overline{A}^{d-j} \cdot \overline{A}^{d+i+j} \quad (0 \leq j \leq d-1).$$

The latter follows from the fact that A (and hence also $\overline{A}$) is generated by elements of degree $\leq d$ (see 3.7). □

Proposition 3.9. The number of binomials in a minimal generating set for the ideal $I \subset A$ is not greater than

$$\binom{2d+V-1}{2d}.$$

Proof. By 3.6, it is sufficient to prove the same statement for a minimal generating set of the ideal $\overline{I} \subset \overline{A}$. By 3.8, the number of minimal generators of $\overline{I} \subset \overline{A}$ is not greater than the dimension of the space of all polynomials in $l-n-1$ variables of degree $\leq 2d$. The latter is not greater than

$$\binom{2d+l-n-1}{2d},$$

because the maximum of this dimension is attained if all $l-n-1$ variables in $\overline{A}$ have degree 1. It remains to apply the inequality $l \leq V+n$ (see 3.7). □

Proof of Theorem 1.3. Let $B_1, \ldots, B_p$ be a minimal generating set of binomials for the ideal I. The number of different variables from $\{X_1, \ldots, X_l\}$ appearing in a binomial B_i is obviously not greater than $2 \deg B_i$

By 3.9, we have $p \leq \binom{2d+V-1}{2d}$. By 3.8, we have $\deg B_i \leq 2d$. Therefore the number of variables from $\{X_1, \ldots, X_l\}$ appearing in at least one binomial relation is not greater than

$$4d\binom{2d+V-1}{2d}.$$

If

$$n \geq 4d\binom{2d+V-1}{2d},$$

then by 3.2

$$|\Delta \cap \mathbb{Z}^n| > 4d\binom{2d+V-1}{2d}$$

and hence there exists a lattice point $x_i \in \Delta \cap \mathbb{Z}^n$ such that the corresponding variable X_i does not appear in any binomial relation $B_1, \ldots, B_p$. Hence, by 3.5, $\Delta = \Pi(\Delta')$ for some $(n-1)$-dimensional lattice polytope.

□

4. Some examples and conjectures

We remark that the estimate for n in Theorem 1.3 is far from being optimal.

Using the complete classification of lattice polytopes with linear h^*-polynomial [**BN**], one can immediately explicitly compute the numbers $C(V,1,n) = C_{h^*}(n)$ where $h^* = h^*_\Delta = 1 + (V-1)t$.

PROPOSITION 4.1. Let $p(n,V)$ be the number of integral solutions of the equation

$$\sum_{i=1}^{n} k_i = V, \quad 0 \leq k_1 \leq \cdots \leq k_n.$$

Then $C(4,1,1) = 1$, $C(4,1,n) = p(n,4) + 1$ (if $n \geq 2$), and $C(V,1,n) = p(n,V)$, (if $V \neq 4, n \geq 1$). In particular, the monotone sequence $C(V,1,n)$ becomes constant for $n \geq V$.

PROOF. By [**BN**], every n-dimensional lattice polytope with linear h^*-polynomial is either a $(n-2)$-fold pyramid of the lattice triangle $T \subset \mathbb{R}^2$ with vertices $(0,0), (2,0), (0,2)$, or a Lawrence prism with heights $k_1, \ldots, k_n$ $(\sum_{i=1}^{b} k_i = V)$. Up to an $AGL(n,\mathbb{Z})$-isomorphism, we can always assume that $0 \leq k_1 \leq \cdots \leq k_n$. $\square$

It is interesting to understand in general which polynomials in $\mathbb{Z}[t]$ can be realised as h^*-polynomials of lattice polytopes. In this connection we propose the following conjecture:

CONJECTURE 4.2. Let $h^* = \sum_{0 \leq i \leq d} h^*_i t^i \in \mathbb{Z}[t]$ be a polynomial of degree d with nonnegative coefficients. If $h^* = h^*_\Delta$ for some n-dimensional lattice polytope Δ, then $h^*(1) = \mathrm{Vol}_{\mathbb{N}}\Delta$ is bounded by some constant depending only on the leading coefficient h^*_d of h^*.

By a theorem of Hensley [**He**], 4.2 is known to be true for $n = d$. Obviously 4.2 is true for arbitrary n if h^* is a linear polynomial. For quadratic polynomials h^*, we expect the following more precise statement:

CONJECTURE 4.3. For any n-dimensional lattice polytope with quadratic h^*-polynomial one has:

$$h^*_1 \leq 3h^*_2 + 4.$$

For $n = 2$ this conjecture is known to be true by a theorem of Scott [**Sc**] (see also [**HS**]).

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